GOSTS Iterated Forcing

Long Iterations and Support

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Refresher

The two-step iteration view takes a preorder $\mathbb{P}$ and a $\mathbb{P}$-name for a preorder $\dot{\mathbb{Q}}$. We then take the "product" of these two:

- Elements of $\mathbb{P} * \dot{\mathbb{Q}}$ are pairs $\langle p, \dot{q} \rangle$ where $\dot{q} \in \mathrm{Cncl}(\dot{\mathbb{Q}})$ canonical name for an element of $\dot{\mathbb{Q}}$:

$$\mathrm{Cncl}(\tau) = \{\sigma : \sigma \text{ is canonical} \land \mathbb{1}^{\mathbb{P}} \Vdash \text{``}\sigma \in \tau\text{''}\}.$$

- We order $\langle p^*, \dot{q}^* \rangle \leqslant \langle p, \dot{q} \rangle$ iff $p^* \leqslant^{\mathbb{P}} p$ and $p^* \Vdash \text{``}\dot{q}^* \dot{\leqslant}^{\dot{\mathbb{Q}}} \dot{q}\text{''}$.

Basically, we go down in each component in the only intelligible way we can.

It's not difficult to show that the trivial preorder $\mathbb{1}$ (with only one element $\emptyset$ ordered by $=$) can be used to frame *any* preorder $\mathbb{P}$ in the ground model as an iteration: $\mathbb{1} * \check{\mathbb{P}} \cong \mathbb{P}$. This view will turn out to be very useful in setting up longer iterations.

This allows us to iterate finitely many times. If we start with $\mathbb{P} \in V$, $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$, we may view $\mathbb{P} * \dot{\mathbb{Q}}$ as built up with two corresponding sequences:

$$\mathbb{P}_0 = \mathbb{1}$$

$$\dot{\mathbb{Q}}_0 = \check{\mathbb{P}} \qquad \mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0 = \mathbb{1} * \dot{\mathbb{Q}}_0$$

$$\dot{\mathbb{Q}}_1 = \dot{\mathbb{Q}} \qquad \mathbb{P}_2 = \mathbb{P}_1 * \dot{\mathbb{Q}}_1 = (\mathbb{1} * \dot{\mathbb{Q}}_0) * \dot{\mathbb{Q}}_1$$

$$\cong \mathbb{P} * \dot{\mathbb{Q}}.$$

In this way, we may view any iteration of a sequence of names of preorders $\langle \dot{\mathbb{Q}}_n : n < N \rangle$, $N < \omega$, as the result of a sequence of actual preorders $\langle \mathbb{P}_n : n \leq N \rangle$ where $\mathbb{P}_0 = \mathbb{1}$ and

$$\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_n = (((\mathbb{1} * \dot{\mathbb{Q}}_0) * \dot{\mathbb{Q}}_1) * \ldots) * \dot{\mathbb{Q}}_n.$$

Formally, this is a terrible monstrosity, but we can dramatically simplify the situation if we reframe what the elements and order of these $\mathbb{P}_n$s look like. Formally, elements of $\mathbb{P}_n$ have the form

$$\langle\langle\langle\langle\emptyset, \dot{q}_0\rangle, \dot{q}_1\rangle, \cdots\rangle, \dot{q}_{n-1}\rangle$$

But really we should view such elements as sequences

$$p : n \to \bigcup_{i<n} \mathrm{Cncl}(\dot{\mathbb{Q}}_i), \quad \text{i.e.} \quad p \in \prod_{i<n} \mathrm{Cncl}(\dot{\mathbb{Q}}_i).$$

In doing so, we also arrive at a much better description of the order $\leqslant_n$ on $\mathbb{P}_n$ (consider instead $n + 1$ for the sake of notation). before, we would have

$$\langle\langle\langle\langle\emptyset, \dot{q}_0^*\rangle, \dot{q}_1^*\rangle, \cdots\rangle, \dot{q}_n^*\rangle \leqslant_n \langle\langle\langle\langle\emptyset, \dot{q}_0\rangle, \dot{q}_1\rangle, \cdots\rangle, \dot{q}_n\rangle$$

iff $\langle\langle\langle\langle\emptyset, \dot{q}_0^*\rangle, \dot{q}_1^*\rangle, \cdots\rangle \leqslant_n \langle\langle\langle\langle\emptyset, \dot{q}_0\rangle, \dot{q}_1\rangle, \cdots\rangle$ and the stronger forces "$\dot{q}_n^* \overset{\dot{\mathbb{Q}}_n}{\leqslant} \dot{q}_n$". As sequences, however, we can rephrase this as saying $p^* \leqslant_{n+1} p$ iff

- $p^* \restriction n \leqslant_n p \restriction n$

- $p^* \restriction n \Vdash$ "$p^*(n) \overset{\dot{\mathbb{Q}}_n}{\leqslant} p(n)$".

which basically says that we go down in each component in the only way that makes sense. Note that this is partly why we start with $\mathbb{1}$: $p \restriction 0 = \emptyset \in \mathbb{1}$ for any p.

Note under this view we have the following properties of these
$\mathbb{P}_n = \langle \mathbb{P}_n, \leq_n, \mathbb{1}_n \rangle$s and $\dot{\mathbb{Q}}_n = \langle\!\langle \dot{\mathbb{Q}}_n, \dot{\leq}'_n, \dot{\mathbb{1}}'_n \rangle\!\rangle$s):

1. Each $\dot{\mathbb{Q}}_n$ is a $\mathbb{P}_n$-name for a preorder;

2. Each $\mathbb{P}_n$ is a preorder with $\mathbb{P}_0 = \mathbb{1}$;

3. Each $p \in \mathbb{P}_n$ is a function $p \in \prod_{i<n} \text{Cncl}(\dot{\mathbb{Q}}_i)$;

4. If $p \in \mathbb{P}_n$ and $m < n$ then $p \restriction m \in \mathbb{P}_m$;

5. If $\dot{\mathbb{Q}}_n$ exists, $p \in \mathbb{P}_{n+1}$ iff $p \restriction n \in \mathbb{P}_n$, $p(n) \in \text{Cncl}(\dot{\mathbb{Q}}_n)$ with
   $p \restriction n \Vdash \text{"} p(n) \in \dot{\mathbb{Q}}_n \text{"}$; and

6. $p^* \leq_{n+1} p$ iff $p^* \restriction n \leq_n p \restriction n$ and $p^* \restriction n \Vdash \text{"} p^*(n) \dot{\leq}'_n p(n) \text{"}$.

From this, the question becomes: what do we do at limit stages? Should
$\mathbb{P}_\omega$ be the *direct limit* of the $\mathbb{P}_n$s? The *inverse limit*? Some other thing
entirely? Sometimes one thing sometimes another? To answer this
question, we need to define and think of the support of our sequences,
which can be thought of as how often the p "does nothing".

In the end, all of the above notions of limit are possible, but they are determined by what kinds of support we allow at limit stages.

Definition

For $\mathbb{P}_n$ and $\langle \dot{\mathbb{Q}}_i : i < n \rangle$ as above, we define the *support* of $p \in \mathbb{P}_n$ to be the set

$$\mathrm{sprt}(p) = \{n \in \mathrm{dom}(p) : \mathbb{1}_n \not\Vdash \text{``}p(n) = \dot{\mathbb{1}}'_n\text{''}\}.$$

Basically we view $p \in \mathbb{P}_n$ as sequences of elements, and the support is just where $p(n)$ isn't $\mathbb{1}^{\dot{\mathbb{Q}}_n}$, i.e. $p(n)$ actually carries some information. To define long iterations, we then restrict our functions p with $\mathrm{dom}(p) = \omega$ to those with support in some $I \subseteq \mathcal{P}(\omega)$.

Definition

Let κ be an ordinal, and $I \subseteq \mathcal{P}(\kappa)$ some collection we think of as allowed supports. A *κ-stage iterated forcing with supports in I* is a pair of sequences $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ and $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ such that for all $\xi \leq \alpha < \kappa$;

❶ Each $\mathbb{P}_\alpha = \langle \mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha \rangle$ is a preorder with $\mathbb{P}_0 = \mathbb{1}$;

❷ Each $\dot{\mathbb{Q}}_\alpha = \langle\langle \dot{\mathbb{Q}}_\alpha, \dot{\leq}'_\alpha, \dot{\mathbb{1}}'_\alpha \rangle\rangle$ is a $\mathbb{P}_\alpha$-name for a preorder;

❸ $\mathbb{1}_\alpha = \langle \dot{\mathbb{1}}'_\xi : \xi < \alpha \rangle$;

❹ Each element of $\mathbb{P}_\alpha$ is a function $p \in \prod_{\xi < \alpha} \mathrm{Cncl}(\dot{\mathbb{Q}}_\xi)$;

❺ If $p \in \mathbb{P}_\alpha$, then $p \restriction \xi \in \mathbb{P}_\xi$;

❻ For $\alpha = \xi + 1$ a successor, we essentially set $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$:
   - $p \in \mathbb{P}_{\xi+1}$ iff $\langle p \restriction \xi, p(\xi) \rangle \in \mathbb{P}_\xi * \dot{\mathbb{Q}}_\xi$,
   - $p^* \leq_{\xi+1} p$ iff $p^* \restriction \xi \leq_\xi p \restriction \xi$ and $p^* \restriction \xi \Vdash$ "$p^*(\xi) \dot{\leq}'_\xi p(\xi)$"; and

❼ For α a limit, we essentially require support in I:
   - $p \in \mathbb{P}_\alpha$ iff $\forall \xi < \alpha \ (p \restriction \xi \in \mathbb{P}_\xi)$ and $\mathrm{sprt}(p) \in I$;
   - $p^* \leq_\alpha p$ iff $\forall \xi < \alpha \ (p^* \restriction \xi \leq_\xi p \restriction \xi)$.

It's not difficult to show that κ-stage iterations yield $\mathbb{P}_\kappa$ as a preorder. Intuitively, each $\mathbb{P}_\alpha$ is the iteration of $\dot{\mathbb{Q}}_\xi$s for $\xi < \alpha$—perhaps better written $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$—with supports in some I which is not indicated with either notation.

I will elect to use $\ast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ for these iterations because I think it is a little more transparent and frees up $\mathbb{P}$. Additionally, it shows that these are dependent on the $\dot{\mathbb{Q}}_\alpha$s. One can see in the above definition that the only restriction of the $\dot{\mathbb{Q}}_\alpha$s is that they are $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-names. The rest of the definition is about defining these $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$s for $\alpha \leq \kappa$.

As another bit of convention, we often refer to $\ast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ as a κ-stage iteration rather than the pair of sequences $\langle \ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi : \alpha \leq \kappa \rangle$ and $\langle \dot{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$, just to save space.

Result

Let κ be an ordinal and $\mathop{\Asterisk}_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ a κ-stage iteration with supports in some $I \subseteq \mathcal{P}(\kappa)$. Therefore $\mathop{\Asterisk}_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ is a preorder.

Proof.

We prove this by induction on κ. For successors, this is clear as we basically defined, for $\kappa = \kappa^* + 1$,

$$\mathop{\Asterisk}_{\alpha<\kappa^*+1} \dot{\mathbb{Q}}_\alpha \cong \left(\mathop{\Asterisk}_{\alpha<\kappa^*} \dot{\mathbb{Q}}_\alpha \right) * \dot{\mathbb{Q}}_{\kappa^*}.$$

So suppose κ is a limit. Transitivity of $\leq_\kappa$ follows by transitivity of the previous $\leq_\alpha$s inductively: $p \leq_\kappa q \leq_\kappa r$ iff $\forall \alpha < \kappa \, (p \restriction \alpha \leq_\alpha q \restriction \alpha \leq_\alpha r \restriction \alpha)$ so $\forall \alpha < \kappa \, (p \restriction \alpha \leq_\alpha r \restriction \alpha)$ and hence $p \leq_\kappa r$. Reflexivity is similarly easy as is showing $\dot{\mathbb{1}}'_\kappa$ is maximal. $\square$

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Introduction to support

To know more about what $\bigstar_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ is for limit α (especially in regards to previous iterations), we must investigate the suppports allowed. Usually, we take these supports to be in an *ideal*, the dual concept to a filter.

Definition

Let X be a set, $I \subseteq \mathcal{P}(X)$. $\emptyset \subsetneq I \subseteq \mathcal{P}(X)$ is an *ideal* iff

- $A \subseteq B \in I$ implies $A \in I$; and

- $A, B \in I$ implies $A \cup B \in I$.

$I = \mathcal{P}(X)$ is called an *improper ideal*, other ideals are proper.

The reason why we want this is that $p \in \bigstar_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ implies $p \restriction \alpha \in \bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$ still with $\mathrm{sprt}(p \restriction \alpha) \subseteq \mathrm{sprt}(p) \in I$, we should have I closed under subsets in this sort of way. It's also advantageous to be able to extend two conditions that "should" be compatible without I getting in the way: if $r \leq p, q$ then we at least have $\mathrm{sprt}(p) \cup \mathrm{sprt}(q) \subseteq \mathrm{sprt}(r)$. So both of these requirements are somewhat natural although not necessary strictly speaking.

There are lots of examples of ideals just as there are lots of examples of filters. In fact, for any filter $F \subseteq \mathcal{P}(X)$, $\{X \setminus A : A \in F\}$ is an ideal (and we can go backwards for proper ideals too). More concrete examples for cardinals $\lambda \leq \kappa$ include the following:

- $\{\emptyset\}$ is an ideal;
- $\{X \subseteq \kappa : |X| \leq \lambda\}$ is an ideal e.g. countable support;
- $\{X \subseteq \kappa : |X| < \lambda\}$ is an ideal e.g. finite support;
- $\{X \subseteq \kappa : X \text{ is bounded in } \kappa\} = \{X \subseteq \kappa : \sup X < \kappa\}$ is an ideal (which is the same as the previous for regular κ);
- $\{Y \subseteq X : x \notin Y\}$ is an ideal for $X \neq \emptyset$, $x \in X$, called a *principal ideal* corresponding to principal filters.

We can show that this preorder $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ in the ground model gives us what we want: any generic G over V breaks down as

$$V = V[G \restriction 0] \subseteq V[G_0] = V[G \restriction 1] \subseteq V[G_0][G_1] = V[G \restriction 2]$$
$$\subseteq \cdots \subseteq V[G \restriction \kappa] = V[G] \ni \{G_\alpha : \alpha < \kappa\}$$

where each G_α is $(\dot{\mathbb{Q}}_\alpha)_{G \restriction \alpha}$-generic over $V[G \restriction \alpha]$ and each $G \restriction \alpha$ is $\ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$-generic over V.

Theorem

Assume the following:

- $\kappa \in \mathrm{Ord} \cap V$;
- $I \subseteq \mathcal{P}(\kappa)$ with $I \in V$ is a non-principal ideal;
- $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha \in V$ is a κ-stage iteration with supports in I;
- G is $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$-generic over V.

Therefore for each $\alpha < \kappa$,

1. $G \restriction \alpha = \{p \restriction \alpha : p \in G\}$ is $\ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$-generic over V; and

2. $G_\alpha = \{p(\alpha)_{G \restriction \alpha} : p \in G\}$ is $(\dot{\mathbb{Q}}_\alpha)_{G \restriction \alpha}$-generic over $V[G \restriction \alpha]$.

Note that this is basically the best that we could hope for: just because we have an increasing sequence of generic extensions

$$V \subseteq V[G_0] \subseteq V[G_0][G_1] \subseteq \cdots$$

doesn't mean that there is some model $V[G]$ at the end with $\{G_n : n < \omega\} \in V[G]$. This is sort of in contrast to the two-step case where $V \subseteq V[G_0] \subseteq V[G_0][G_1]$ yields $V[G_1] = V[G_0 * G_1]$ as a generic extension of a preorder in V.

We can sort-of trivially see the issue if we have, say, a countable transitive model V with some $\langle \gamma_n : n < \omega \rangle$ cofinal in Ord $\cap V$ and we continually force with $\mathrm{Col}(\aleph_0, \gamma_n)$ at each stage. The nth generic G_n will have

$$V \subseteq V[G_0] \subseteq V[G_0][G_1] \subseteq \cdots$$

but there will not be any extension W modelling ZFC with $V \subseteq W$ and $\{G_n : n < \omega\} \in W$ because such a model would have every ordinal as countable, which is an issue.

This is sort-of cheating, though, since $\langle \gamma_n : n < \omega \rangle$ isn't definable over V so we can't even define the iteration.

There is a more subtle argument, however, that shows we can't just amalgamate generics and get another generic extension above all the previous ones in general. Again, assume V is countable and let $r : \omega \to \mathrm{Ord} \cap V$ be a bijection. If we force with $\mathrm{Add}(\aleph_0, 1)$ at each stage, we can pretty easily arrange that the nth generic G_n has its first entry as $r(n)$. Thus we'd get $V \subseteq V[G_0] \subseteq V[G_0][G_1] \subseteq \cdots$, but any extension W modelling ZFC with $V \subseteq W$ and $\{G_n : n < \omega\} \in W$ would be able to reconstruct r and thus know the countability of $\mathrm{Ord} \cap V = \mathrm{Ord} \cap W$, which is an issue.

So we can't always amalgamate generics even in principle unlike with two-step iterations. But we do at least get the above result.

Result

❶ $G \upharpoonright \alpha = \{p \upharpoonright \alpha : p \in G\}$ is $\bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-generic over V if G is $\bigstar_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$-generic over V.

Proof of (1).

First we show $G \upharpoonright \alpha$ is a filter.

- Since G is a filter and $p^* \leq_\kappa p$ implies $p^* \upharpoonright \alpha \leq_\alpha p \upharpoonright \alpha$, it follows that any two elements of $G \upharpoonright \alpha$ are compatible.

- $G \upharpoonright \alpha$ is closed upward since if $p \upharpoonright \alpha \in G \upharpoonright \alpha$ with $p \upharpoonright \alpha \leq_\alpha q$ then we can translate q to $\bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$ just by adding a tail of $\dot{\mathbb{1}}'_\xi$s:

  $q' = q ^\frown \langle \dot{\mathbb{1}}'_\xi : \alpha \leq \xi < \kappa \rangle$ so that $p \leq_\kappa q'$ and $q' \upharpoonright \alpha = q \in G \upharpoonright \alpha$, as desired.

Proof of the iterated forcing theorem

Result

❶ $G \restriction \alpha = \{p \restriction \alpha : p \in G\}$ is $\Asterisk_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-generic over V if G is $\Asterisk_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$-generic over V.

Proof of (1).

To show genericity, let $D \subseteq \Asterisk_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$ be dense in V. Let $D' = \{q \in \Asterisk_{\xi < \kappa} \dot{\mathbb{Q}}_\xi : q \restriction \alpha \in D\}$. This D' is dense in $\Asterisk_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ since if $p \in \Asterisk_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ is arbitrary, then $p \restriction \alpha$ has an extension $p^* \in D$. We have

$$\mathrm{sprt}(p^* {}^\frown (p \restriction [\alpha, \kappa)) \subseteq \mathrm{sprt}(p) \cup \mathrm{sprt}(p^*) \in I,$$

and therefore we have an actual condition $p^* {}^\frown (p \restriction [\alpha, \kappa))$ in $\Asterisk_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ and by construction also in D'. Therefore $G \cap D' \neq \emptyset$ and any $p \in G \cap D'$ has $p \restriction \alpha \in (G \restriction \alpha) \cap D \neq \emptyset$, showing genericity. $\qquad\square$

Proof of the iterated forcing theorem

Result

❷ $G_\alpha = \{p(\alpha)_{G \restriction \alpha} : p \in G\}$ *is* $(\dot{\mathbb{Q}}_\alpha)_{G \restriction \alpha}$*-generic over* $V[G \restriction \alpha]$.

Proof of (2).

Write $\mathbb{Q}_\alpha$ for $(\dot{\mathbb{Q}}_\alpha)_{G \restriction \alpha}$. Firstly, G_α is a filter:

- For $p(\alpha)_{G \restriction \alpha}, q(\alpha)_{G \restriction \alpha} \in G_\alpha$, we find a common extension $r \leqslant_\kappa p, q \in G$ giving that $r \restriction (\alpha + 1) \leqslant_{\alpha+1} p \restriction (\alpha + 1), q \restriction (\alpha + 1)$ and therefore $r \restriction \alpha \Vdash$ "$r(\alpha) \dot{\leqslant}'_\alpha p(\alpha), q(\alpha)$" with therefore $r \restriction \alpha \in G \restriction \alpha$. It follows that $r(\alpha)$ is a common extension to $p(\alpha)_{G \restriction \alpha}$ and $q(\alpha)_{G \restriction \alpha}$.

- For upward closure, $p^*(\alpha)_{G \restriction \alpha} \leqslant'_\alpha p$ with $p^* \in G$ yields

$$q = (p^* \setminus \{\langle \alpha, p^*(\alpha)\rangle\}) \cup \{\langle \alpha, \dot{p}\rangle\} \in \text{\Large✳}_{\xi < \kappa} \dot{\mathbb{Q}}_\xi,$$

  and $p^* \leqslant_\kappa q \in G$ and therefore $q(\alpha)_{G \restriction \alpha} = \dot{p}_{G \restriction \alpha} = p \in G_\alpha$.

Result

❷ $G_\alpha = \{p(\alpha)_{G \restriction \alpha} : p \in G\}$ *is* $(\dot{\mathbb{Q}}_\alpha)_{G \restriction \alpha}$*-generic over* $V[G \restriction \alpha]$.

Proof of (2).

To show genericity of G_α, let $D \subseteq \mathbb{Q}_\alpha$ be dense in $V[G \restriction \alpha]$. There is some condition $p_D \in G$ with $p_D \restriction \alpha \Vdash$ "$\dot{D}$ is dense in $\dot{\mathbb{Q}}_\alpha$" (can get $p_D = \mathbb{1}_\alpha$ if $\dot{D}$ is chosen well enough). We get (by similar arguments as before) that

$$D' = \{q \leqslant_\kappa p_D : p_D \restriction \alpha \Vdash \text{"}q(\alpha) \in \dot{D}\text{"}\}$$

is dense below p_D in $\bigstar_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$. In particular, there is some $q \in G \cap D'$ where then $q(\alpha)_{G \restriction \alpha} \in G_\alpha \cap D \neq \emptyset$. This shows genericity. □

The role of support

One result of the above theorem is that the kind of support we take doesn't impact that we end up with a generic extension beyond the extensions of the previous iterations But this alone doesn't tell us much about what else we might have inadvertently added. Support, in telling us what the iterations $\ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$ look like for limit $\alpha \leq \kappa$, then give us information about $V[G \restriction \alpha]$ according to similar properties we've already studied (e.g. being κ-cc, $< \kappa$-closed, and so on). The most common kinds of support are all amalgamations of the following:

Definition

Let κ be an ordinal, $I \subseteq \mathcal{P}(\kappa)$, and $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ a κ-stage iteration with supports in I. We say this is a

- *finite support* iteration iff $I = \{X \subseteq \kappa : |X| < \aleph_0\}$;
- *bounded support* iteration iff $I = \{X \subseteq \kappa : \sup X < \kappa\}$;
- *full support* iteration iff $I = \mathcal{P}(\kappa)$;
- *countable support* iteration iff $I = \{X \subseteq \kappa : |X| \leq \aleph_0\}$.

The role of support

We will only look at finite, bounded, and full support here. Countable support is important for preservation of being "proper". The others are important for their preservation properties but also for how nice we can describe the limit preorder $\bigstar_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ model-theoretically.

In particular,

- Bounded support at stage κ corresponds to taking the direct limit of previous iterations;
- Finite support corresponds to taking direct limits of previous iterations at every limit stage;
- Full support corresponds to taking inverse limits of previous iterations at every limit stage.
- There's a notion of *unbounded* or prehaps *supremum* support at stage κ corresponding to taking the inverse limit of previous iterations at that stage. (I'm unaware of a standard name for this.)

Bounded support at stage κ means $I \cap \mathcal{P}(\kappa) = \bigcup_{\alpha < \kappa} I \cap \mathcal{P}(\alpha)$.
Supremum support at stage κ I am using to mean
$I \cap \mathcal{P}(\kappa) = \{\bigcup x : x \subseteq \bigcup_{\alpha < \kappa} I \cap \mathcal{P}(\alpha)\}$, basically unbounded unions of previously allowed supports.

Direct limits

Informally, the direct limit of a bunch of models is just the least amount we need to capture all the information of the previous models. This means it's sort of a least upper bound when it comes to embeddings.

If $\mathcal{A}$ is a set of models, $\mathcal{F}$ is a set of upward directed embeddings, the direct limit $\text{dir lim}_{\mathcal{F}} \mathcal{A}$ looks like the following (where all maps commute).
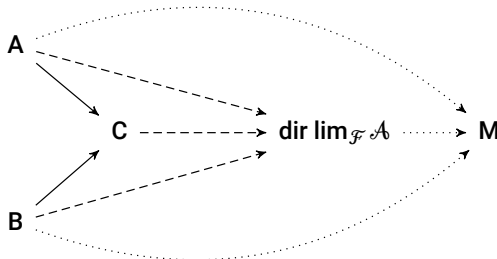
(Solid lines are in $\mathcal{F}$, dashed lines are direct limit embeddings, dotted are to any other supposed direct limit.)

It's not too bad to show that direct limits always exist and can be constructed as the disjoint union of the models modulo equivalence from the embeddings: $x \approx y$ iff there is are embeddings sending x and y to the same place. So the universe is $\operatorname{dir} \lim_{\mathcal{F}} \mathcal{A} = \left\{ [x]_{\approx} : x \in \bigsqcup_{A \in \mathcal{A}} A \right\}$. Relations and functions are similarly interpreted: R is true of $\vec{x}$ iff we can send every entry of $\vec{x}$ to the same place and have R be true there.

Formally, we need $\langle \mathcal{A}, \mathcal{F} \rangle$ needs to be a upward-directed system of embeddings. This is easily accomplished since appending $\dot{\mathbb{1}}'_\xi$s gives embeddings into later iterations.

Result

Let $\bigast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ be a κ-stage iteration with supports in some ideal $I \subseteq \mathcal{P}(\kappa)$. Therefore for each $\alpha < \beta \leq \kappa$;

- *There is an embedding $i_{\alpha,\beta} : \bigast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi \to \bigast_{\xi < \beta} \dot{\mathbb{Q}}_\xi$ defined by*

$$\iota_{\alpha,\beta}(p) = p ^\frown \langle \dot{\mathbb{1}}'_\xi : \alpha \leq \xi < \beta \rangle.$$

- *Moreover, for incompatibility, $p \perp q$ iff $\iota_{\alpha,\beta}(p) \perp \iota_{\alpha,\beta}(q)$.*

Note: as preorders are $\{\leqslant, \mathbb{1}\}$-models, homomorphisms and embeddings just need to be order preserving (and maximal element preserving), but it's often useful to define embeddings and homomorphisms to also preserve incompatibility as above. Doing this doesn't make a difference for the direct limit and it will be the same in either case. For the inverse limit, however, homomorphisms should *not* preserve incompatibility.

With these $\iota_{\alpha, \beta}$ embeddings, everything works together and the natural "least upper bound" is just consists of conditions that eventually end in a tail of $\dot{\mathbb{1}}'_\xi$s. This means taking *bounded* support in our iterations. And it turns out that this works.

Theorem

Let $\bigstar_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ be a non-trivial, κ-stage iteration with supports in some ideal $I \subseteq \mathcal{P}(\kappa)$ where κ is a limit. Therefore, $\bigstar_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ is the direct limit of previous iterations iff every $x \in I$ is bounded in κ, meaning $I \cap \mathcal{P}(\kappa) = \bigcup_{\alpha < \kappa} I \cap \mathcal{P}(\alpha)$.

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Theorem

$\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ *is the direct limit of previous iterations iff every $x \in I$ is bounded in κ.*

Proof.

The ($\rightarrow$) direction is pretty clear: any $p \in \ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ with unbounded support is not in the image of any $\iota_{\alpha,\kappa}$ for $\alpha < \kappa$ and hence $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha \neq \bigcup_{\alpha<\kappa} \iota_{\alpha,\kappa}" \ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$, which is necessary for being the direct limit.

The ($\leftarrow$) direction isn't particularly enlightening and we do not prove it here (see notes I've been writing). □

Thus finite support iterations always take direct limits is the only (natural) kind of support where this happens.

Corollary

Let $\bigstar_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ be a κ-stage, finite support iteration. Therefore $\bigstar_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$ is the direct limit of previous iterations for each limit α. Moreover, any non-trivial κ-length iteration taking direct limits at every limit stage has finite support.

Proof.

As finite support is bounded in every limit, we always take the direct limit by the previous theorem. To see that finite support iterations are the only iterations with this property, we proceed by induction on limit $\alpha \leq \kappa$.

For $\alpha = \omega$, this is clear: bounded support in ω requires finite support below ω. For $\alpha > \omega$, any $p \in \bigstar_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$ has $p = \iota_{\beta,\alpha}(p\restriction\beta)$ for some $\beta < \alpha$. But then the support of both is inductively finite (even if β is a successor, it's only finitely many iterations above a limit): $\mathrm{sprt}(p) = \mathrm{sprt}(p\restriction\beta) \in I \cap \mathcal{P}(\beta) \subseteq \{x \subseteq \beta : x \text{ is finite}\}$.

Corollary

Let $\textasteriskcentered_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ be a κ-stage, finite support iteration. Therefore $\textasteriskcentered_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$ is the direct limit of previous iterations for each limit α. Moreover, any non-trivial κ-length iteration taking direct limits at every limit stage has finite support.

Proof.

Additionally, we must have $I \cap \mathcal{P}(\beta) \supseteq \{x \subseteq \beta : x \text{ is finite}\}$. This is because at successor stages of the iteration, we allow ourselves to extend the support by one element which eventually yields any finite number of elements all of which must be allowed at stage α as we're taking the direct limit. $\square$

The benefit of things like direct limits is preservation of chain conditions. In general we have the following.

Theorem

Let λ be a limit ordinal and κ a cardinal. Let $\ast_{\alpha<\lambda} \dot{\mathbb{Q}}_\alpha$ be a λ-length iteration with support in some ideal $I \subseteq \mathcal{P}(\lambda)$. Suppose

❶ $\ast_{\alpha<\lambda} \dot{\mathbb{Q}}_\alpha$ *is the direct limit of previous iterations;*

❷ $\ast_{\xi<\alpha} \dot{\mathbb{Q}}_\alpha$ *is κ-cc for each $\alpha < \lambda$; and*

❸ $\mathrm{cof}(\lambda) = \kappa$ *implies*
$\{\alpha < \lambda : \ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$ *is the direct limit of previous iterations$\}$ is stationary in λ.*

Therefore $\ast_{\alpha<\lambda} \dot{\mathbb{Q}}_\alpha$ is κ-cc.

We will prove only the case of $\kappa = \aleph_1$ (i.e. ccc) with finite support iterations. (See the notes for the general proof.)

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Direct limits

We will prove only the case of $\kappa = \aleph_1$ (i.e. ccc) with finite support iterations. (See the notes for the general proof.)

Theorem

Let $\ast_{\alpha < \kappa} \dot{Q}_\alpha$ be a κ-stage, finite support iteration such that for all $\alpha \leq \kappa$, $\mathbb{1}_\alpha \Vdash$ "$\dot{Q}_\alpha$ is ccc". Therefore $\ast_{\xi < \alpha} \dot{Q}_\xi$ is ccc for every $\alpha \leq \kappa$.

Proof.

Proceed by induction on $\alpha \leq \kappa$. Clearly for $\alpha = 0$, $\ast_{\xi < 0} \dot{Q}_\xi = \mathbb{1}$ is ccc. Inductively the successor case follows by the previous talk: $\ast_{\xi < \alpha} \dot{Q}_\xi$ is ccc and $\mathbb{1}_\alpha \Vdash$ "$\dot{Q}_\alpha$ is ccc" implies $\ast_{\xi < \alpha+1} \dot{Q}_\xi \cong \ast_{\xi < \alpha} \dot{Q}_\xi \ast \dot{Q}_\alpha$ is ccc.

So suppose α is a limit but the result fails: $\mathcal{A}$ is an uncountable antichain. By the Δ-system lemma, since all supports are finite, we may assume all the conditions of $\mathcal{A}$ have the same intersection: $p, q \in \mathcal{A}$ implies $\mathrm{sprt}(p) \cap \mathrm{sprt}(q) = r$ for some finite $r \subseteq \alpha$. But then incompatibility is due to a disagreement in $r \subseteq 1 + \max r$: $\{p \restriction (1 + \max r) : p \in \mathcal{A}\}$ is an uncountable antichain of $\ast_{\xi < 1 + \max r} \dot{Q}_\xi$, contradicting the inductive hypothesis. $\quad\square$

Inverse Limits

Inverse limits appear much less in logic than the direct limit does. It's basically the greatest lower bound of a (downward directed) set of models as below:
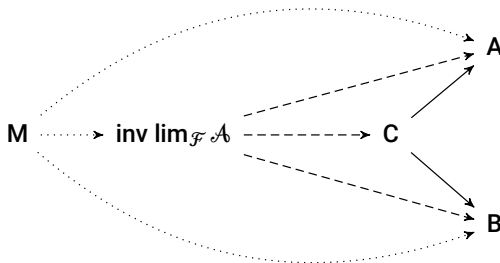
The basic idea isn't that **inv lim**$_{\mathcal{F}}\mathcal{A}$ *embeds* into everything but instead that it *projects* onto everything. In essence, it's the minimum amount of elements that are able to generate everything else by projections (i.e. homomorphisms). This is how the inverse limit can be "bigger" than the models of $\mathcal{A}$.

Inverse limits

The formal definition of an inverse limit talks about $\langle \mathcal{A}, \mathcal{F} \rangle$ being a *projective system of homomorphisms* which basically means $\mathcal{F}$ consists of homomorphisms between the models of $\mathcal{A}$ that all play nicely with composition, and the models are downward directed (by homomorphisms in $\mathcal{F}$).

The inverse limit is just something that projects (playing nicely with composition in $\mathcal{F}$) to all of the models of $\mathcal{A}$ and any other such model has a unique projection to the inverse limit playing nicely with the inverse limit's projections and those in $\mathcal{F}$. In general, we always get that the inverse limit exists (although its theory might be significantly different from the models of $\mathcal{A}$). $\mathrm{inv}\lim_{\mathcal{F}} \mathcal{A}$ is equal to

$$= \left\{ x \in \prod_{\mathbf{A} \in \mathcal{A}} A : \forall \mathbf{A}, \mathbf{B} \in \mathcal{A} \ (f_{\mathbf{A},\mathbf{B}} \in \mathcal{F} \rightarrow f_{\mathbf{A},\mathbf{B}}(x(\mathbf{A})) = x(\mathbf{B})) \right\}.$$

Relation interpretations have $R(\vec{x})$ iff for all $\mathbf{A} \in \mathcal{A}$, $R^{\mathbf{A}}(\vec{x}(\mathbf{A}))$. Basically, we take the product of everything and take the elements that play nicely with the projections.

Inverse limits

Projections for us are just restrictions. So "playing nicely" with these restrictions just means that all of the restrictions are in previous iterations.

Result

Let $\circledast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ be a κ-stage iteration. Therefore, for each $\alpha < \beta \leq \kappa$, the restriction map $\pi_{\beta, \alpha} = (p \mapsto p \restriction \alpha)$ is a homomorphism from $\circledast_{\xi < \beta} \dot{\mathbb{Q}}_\xi$ to $\circledast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$.

This motivates the following definition or result depending on how you frame things.

Definition

We say $\circledast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ is the *inverse limit of previous iterations* iff

$$\circledast_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha = \left\{ p \in \prod_{\alpha < \kappa} \mathrm{Cncl}(\dot{\mathbb{Q}}_\alpha) : \forall \alpha < \kappa \left(x \restriction \alpha \in \circledast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi \right) \right\}.$$

In the previous sense, $x \in \mathrm{inv} \lim_{\mathscr{F}} \mathcal{A}$ took the form $x = \langle p \restriction \alpha : \alpha < \kappa \rangle$ so we have just identified x with $\bigcup_{\alpha < \kappa} x(\alpha) = \bigcup_{\alpha < \kappa} p \restriction \alpha$.

Inverse limits

It should then make sense why supremum support gives inverse limits.

Theorem

Let $\underset{\alpha < \kappa}{\bigast} \dot{\mathbb{Q}}_\alpha$ *be a non-trivial κ-stage iteration with support in a non-principal ideal $I \subseteq \mathcal{P}(\kappa)$. Therefore $\underset{\xi < \alpha}{\bigast} \dot{\mathbb{Q}}_\xi$ is the inverse limit of previous iterations iff we use supremum support at stage α:*

$$\{x \in I \cap \mathcal{P}(\alpha) : \sup x = \alpha\} = \{x \in \mathcal{P}(\alpha) : \sup x = \alpha \land \forall \beta < \alpha \, (x \cap \beta \in I)\}$$

Proving this isn't particularly interesting since it relies on the fine details of defining the inverse limit (see the notes). The basic idea is that if every restriction of p is in previous iterations, then we should have $\mathrm{sprt}(p \restriction \beta) \subseteq \beta$ is in I for each $\beta < \mathrm{dom}(p)$ so that $\mathrm{sprt}(p) = \bigcup_{\beta < \mathrm{dom}(p)} \mathrm{sprt}(p \restriction \beta)$. And going the other direction lets us *build up* a p by building up the support and being the inverse limit requires the limit support to be in the poset.

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Full support then always takes inverse limits.

Corollary

Let $\ast_{\alpha<\kappa} \dot{\mathbb{Q}}_\alpha$ be a κ-stage, full support iteration. Therefore $\ast_{\xi<\alpha} \dot{\mathbb{Q}}_\xi$ is the inverse limit of previous iterations for each limit α. Moreover, full support is the only non-principal ideal with this property.

Proof.

Since every support is in I, we immediately get that the limit of supports in I is also in I and therefore we are using supremum support and thus take the inverse limit at every limit stage.

Suppose we take inverse limits at every limit stage and our support is in I. Since I is closed under finite unions and is non-principal, all finite subsets of κ are in I. Since we're taking the inverse limit at stage ω,

$$\{x \in I \cap \mathcal{P}(\omega) : \sup x = \omega\} = \{x \in \mathcal{P}(\omega) : \sup x = \omega \wedge \forall n < \omega \, (x \cap n \in I)\}$$

meaning that all infinite subsets of ω are in I in addition to all finite subsets: $I \cap \mathcal{P}(\omega) = \mathcal{P}(\omega)$. $\qquad \square$

Corollary

Full support is the only non-principal ideal that takes inverse limits at every stage.

Proof.

Inductively, suppose we have $I \cap \mathcal{P}(\beta) = \mathcal{P}(\beta)$ for all limit $\beta < \alpha$. It follows that $I \cap \mathcal{P}(\beta) = \mathcal{P}(\beta)$ for all $\beta < \alpha$ including successors. Taking the inverse limit at stage α, since all bounded subsets of α are in I, it follows as with ω that all unbounded subsets of α are also in I ($X \subseteq \alpha$ has $X = \bigcup_{\beta < \alpha} X \cap \beta$ with $X \cap \beta \in I$) and therefore $I \cap \mathcal{P}(\alpha) = \mathcal{P}(\alpha)$ for all limit α. □

Technically, it's still possible to take the inverse limit at every stage without full support if we allow trivial preorders at certain stages α and remove some elements containing α. But these supports are somewhat *ad hoc*. Natural examples of support with have I being a non-principal ideal.

Inverse limits

Inverse limits, allowing lots of support, usually work well with properties stating the existence of certain elements below others. For example, as with direct limits and being κ-cc, we have a pretty long theorem detailing sufficient conditions for being $< \kappa$-closed.

Theorem

Let $\bigast_{\alpha < \lambda} \dot{\mathbb{Q}}_\alpha$ be a λ-stage iteration with support in some ideal $I \subseteq \mathcal{P}(\lambda)$ such that

- *inverse limits are taken at every limit stage $\alpha \leq \lambda$ with $\mathrm{cof}(\alpha) < \kappa$;*
- *we take either direct or inverse limits at all other stages;*
- *$\mathbb{1}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is $< \check{\kappa}$-closed" for each $\alpha < \lambda$.*

Therefore $\bigast_{\alpha < \lambda} \dot{\mathbb{Q}}_\alpha$ is $< \kappa$-closed.

As with the κ-cc theorem before, it's easiest to show just in the case of full support, but the general version isn't terribly worse conceptually. Note that the use of canonical names here is needed (or any similar kind of name).

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Corollary

Let $\bigstar_{\alpha<\lambda} \dot{\mathbb{Q}}_\alpha$ be a full support λ-stage iteration. Suppose $\mathbb{1}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is $< \check{\kappa}$-closed" for each $\alpha < \lambda$. Therefore $\bigstar_{\alpha<\lambda} \dot{\mathbb{Q}}_\alpha$ is $< \kappa$-closed.

Proof.

Proceed by induction on λ. $\lambda = 0$ is trivial, λ a successor was shown previously. For λ a limit, let $\langle p_\eta : \eta < \theta \rangle$ be a $\leq_\lambda$-decreasing sequence of length $\theta < \kappa$, meaning $\langle p_\eta \restriction \alpha : \alpha < \lambda \rangle$ is $\leq_\alpha$-decreasing for each $\alpha < \theta$. We will define p below each p_η by defining $p \restriction \alpha$ for each $\alpha \leq \lambda$.

Firstly, define $p \restriction 0 = \emptyset$. For $p \restriction \alpha$ defined thus far, we inductively have

$$p \restriction \alpha \in \bigstar_{\xi<\alpha} \dot{\mathbb{Q}}_\xi \text{ and } p \restriction \alpha \leq_\alpha p_\eta \restriction \alpha \text{ for all } \eta < \theta.$$

Since $p \restriction \alpha$ is below all the others, we get

$$p \restriction \alpha \Vdash \text{"}\langle p_\eta : \eta < \check{\theta} \rangle \text{ is decreasing} \wedge \check{\theta} < \check{\kappa} \wedge \dot{\mathbb{Q}}_\alpha \text{ is } < \check{\kappa}\text{-closed"}$$

Corollary

Let $\bigstar_{\alpha < \lambda} \dot{\mathbb{Q}}_\alpha$ be a full support λ-stage iteration. Suppose $\mathbb{1}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is $< \check{\kappa}$-closed" for each $\alpha < \lambda$. Therefore $\bigstar_{\alpha < \lambda} \dot{\mathbb{Q}}_\alpha$ is $< \kappa$-closed.

Proof.

$$p \restriction \alpha \Vdash \text{"} \langle p_\eta : \eta < \check{\theta} \rangle \text{ is decreasing} \wedge \check{\theta} < \check{\kappa} \wedge \dot{\mathbb{Q}}_\alpha \text{ is } < \check{\kappa}\text{-closed"}$$

and therefore $p \restriction \alpha$ forces some name below all of the $p_\eta(\alpha)$s. Using canonical names, we can find an actual instance $p(\alpha)$ where $p \restriction \alpha ^\frown p(\alpha) \in \bigstar_{\xi < \alpha + 1} \dot{\mathbb{Q}}_\xi$. This defines $p \restriction \alpha + 1$. For limit α, $p \restriction \alpha = \bigcup_{\beta < \alpha} p \restriction \beta$ so this defines $p = p \restriction \lambda$ everywhere. Since we take full support, we automatically get $p \in \bigstar_{\alpha < \lambda} \dot{\mathbb{Q}}_\alpha$ and $p \leq_\lambda p_\eta$ for all $\eta < \theta$. $\square$

Defining other supports

The main idea here is partly to motivate more complicated supports. Elsewhere in the literature, supports may be defined by where direct or inverse limits are taken. Translating this into bounded and supremum support tells more concretely what the elements of the iteration look like. The translation is more-or-less unnecessary, and it's more intuitive to think of the preorders in terms of their structure and how they relate to previous iterations. The more important translation goes in the other direction: taking things in terms of support and translating this to understand the limit stages of the iterations.

For example Easton support[1] is usually defined by something like

$$I = \{x \subseteq \kappa : \forall \lambda < \kappa \ (\lambda \text{ is regular} \rightarrow |\text{Reg} \cap x \cap \lambda| < \lambda)\},$$

where $\text{Reg} = \{\theta \in \text{Ord} : \theta \text{ is regular}\}$. On the surface, this is hard to unpack. But we can break it down by thinking about what happens at each limit stage.

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[1]Usually, with easton support we force with trivial preorders at non-regular cardinal stages, so some definitions might differ slightly from this (e.g. Jech).

$$I = \{x \subseteq \kappa : \forall \lambda < \kappa \ (\lambda \text{ is regular} \rightarrow |\text{Reg} \cap x \cap \lambda| < \lambda)\}.$$

We really care about $x \cap \lambda$ for $\lambda < \kappa$, i.e. $\mathcal{P}(\lambda)$:

- If λ isn't regular, we take full support below λ.

- If λ is regular because it's a successor cardinal, we also take full support below λ.

- If λ is regular but a limit cardinal (i.e. weakly inaccessible), then the support must be smaller than the cofinality of λ, i.e. the support must be *bounded*.

So Easton support can also be described as taking direct limits at inaccessible stages and inverse limits everywhere else.

This idea is also useful in *factoring* because often we want to break up an iteration $\text{\Large$*$}_{\alpha < \kappa} \dot{\mathbb{Q}}_\alpha$ to be $\text{\Large$*$}_{\xi < \alpha} \dot{\mathbb{Q}}_\xi * \text{\Large$*$}_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$ and knowing information about the initial segments can help.

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- In the end, we'd like to have

$$\operatorname*{\text{\Large ✱}}_{\xi<\kappa} \dot{\mathbb{Q}}_\xi = \operatorname*{\text{\Large ✱}}_{\xi<\alpha} \dot{\mathbb{Q}}_\xi * \operatorname*{\text{\Large ✱}}_{\alpha\leq\xi<\kappa} \dot{\mathbb{Q}}_\xi.$$

In essence, viewing $V[G] = V[G\restriction\alpha][G\restriction[\alpha,\kappa)]$.

- The issue with this is somewhat technical, but results basically show that the technical issues are irrelevant and this over-simplified idea is essentially correct in every meaningful way.

Factoring

The natural way of defining the tail iteration such that $V[G] = V[G \restriction \alpha][G \restriction [\alpha, \kappa)]$ is just to take the full iteration, and restrict the elements: for $p \in \bigstar_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$, consider the tail $p \restriction [\alpha, \kappa)$.

Unfortunately, this approach has some name issues: $p \restriction [\alpha, \kappa)$ isn't a $\bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-name. So we just translate it in the natural way (don't translate it at all).

Definition

Let κ be an ordinal, $\alpha < \kappa$, and $\bigstar_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ a κ-stage iteration. Define the $\bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-name

$$\bigstar_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi = \left\{ (p \restriction [\alpha, \kappa))^\smile : p \in \bigstar_{\xi < \kappa} \dot{\mathbb{Q}}_\xi \right\}.$$

We order these elements with the $\bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-name for a preorder

$$\dot{\le}_\alpha^\kappa = \left\{ \langle \langle\!\langle q, r \rangle\!\rangle, p \rangle : p \in \bigstar_{\xi < \alpha} \dot{\mathbb{Q}}_\xi \wedge p ^\frown q \le_\kappa p ^\frown r \right\}.$$

All of this is just a very formal way of saying that we're merely restricting the conditions' domains, and order things in the same way as before: in a generic extension by the initial iteration,

$$q \preccurlyeq_\alpha^\kappa r \quad \text{iff} \quad \exists p \in G \restriction \alpha \ (p \frown q \leqslant_\kappa p \frown r).$$

This pretty easily ensures that $\dot{\preccurlyeq}_\alpha^\kappa$ is forced to be a preorder (and that $\mathbb{1}_\kappa \restriction [\alpha, \kappa)$ is maximal).

There are two main properties we're interested in with the tail iteration that aren't technically true: for $G \restriction \alpha \ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi$-generic over V,

① $\ast_{\xi < \kappa} \dot{\mathbb{Q}}_\xi = \ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi \ast \star_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$;

② $\star_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$ is an iteration in $V[G \restriction \alpha]$;

Nevertheless, we get two properties that are good enough and justify the intuition:

① $\ast_{\xi < \kappa} \dot{\mathbb{Q}}_\xi$ is forcing equivalent to $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_\xi \ast \star_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$;

② $\star_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$ is *isomorphic* to a $(\kappa - \alpha)$-stage iteration in $V[G \restriction \alpha]$;

Proving isn't difficult by the dense homomorphism $f(p) = \langle p \restriction \alpha, p_{\alpha, \kappa} \rangle$ where $p_{\alpha, \kappa}$ is a canonical name for $(p \restriction [\alpha, \kappa))^{\smile}$.

Proving the second, however, is incredibly complicated and technical, requiring recursively defined isomorphisms and translating names. The result should really be thought of as trivial, however: these technical obstacles are not conceptual ones. (See notes for proofs of both.)

I thought about including overviews of the proofs, but they are so technical and god-damn boring there's just no point. The main problem for (2) is that each $\dot{\mathbb{Q}}_\beta$ is a $\text{\Large{∗}}_{\xi < \beta} \dot{\mathbb{Q}}_\xi$-name, not a $\text{\Large{∗}}_{\xi < \alpha} \dot{\mathbb{Q}}_\xi * \text{\Large{∗}}_{\alpha \leq \xi < \beta} \dot{\mathbb{Q}}_\xi$-name.

The results are pretty useful, though, and mostly unstated. For example, we often use these with elementary embeddings because it allows us to use properties of the critical point: if $\text{crit}(j) = \kappa$, $j\left(\text{\Large{∗}}_{\xi < \kappa} \dot{\mathbb{Q}}_\xi\right)$ can be thought of as

$$\underset{\xi < \kappa}{\text{\Large{∗}}} \dot{\mathbb{Q}}_\xi * \underset{\kappa \leq \xi < j(\kappa)}{\text{\Large{∗}}} \dot{\mathbb{P}}_\xi$$

(so long as the $\dot{\mathbb{Q}}_\xi$s are small enough etc.)

As a result, knowing properties of these tail iterations from properties of the whole iteration can be informative.

- If $\bigstar_{\xi<\kappa} \dot{\mathbb{Q}}_\xi$ is the direct limit of previous iterations, then (in $V[G \restriction \alpha]$, the tail iteration $\bigstar_{\alpha \le \xi < \kappa} \dot{\mathbb{Q}}_\xi$ is the direct limit of $\bigstar_{\alpha \le \xi < \beta} \dot{\mathbb{Q}}_\xi$ for $\alpha < \beta < \kappa$.

- We don't get the same for inverse limits, unfortunately, but there are some partial results that also require introducing more terminology and involve uninteresting, technical proofs (see Baumgartner's survey paper).